1.8

Complex Zeros and the Fundamental Theorem of Algebra

<u>Definition:</u> A complex number is any number that can be written in the form *a + bi* where *a and b* are real numbers. The real number *a* is the real part, the real number *b* is the imaginary part, and *a + bi* is the standard form.

A real number *a* is the complex number *a + 0i*, so all real numbers are also complex numbers.

<u>Addition and Subtraction of Complex Numbers</u>

Sum: (*a + bi*) + (*c + di*) = (*a + c*) + (*bi + di*)

Difference: (*a + bi*) - (*c + di*) = (*a - c*) + (*bi - di*)

Additive Inverse: (*a + bi*) is -(*a + bi*) = *-a – bi*

<u>Complex Conjugate</u>:

The complex conjugate of the complex number $z = a + bi$, is $\bar{z} = \overline{a+bi} = a - bi$.

<u>Complex Solutions of Quadratic Equations</u>:

Discriminant is $b^2 - 4ac$ of $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac < 0$ then the root (solution) is a complex number

If $b^2 - 4ac > 0$ then there are two real roots (solutions)

If $b^2 - 4ac = 0$ then there is one real root (solution)

If $b^2 - 4ac < 0$ then there are two complex conjugate roots

<u>Fundamental Theorem of Algebra</u>: A polynomial function of degree n has n complex zeros (real and nonreal). Some of the zeros may be repeated.

<u>Linear Factorization Theorem:</u> If $f(x)$ is a polynomial function of degree $n > 0$, then $f(x)$ has precisely n linear factors and

$$f(x) = a(x - z_1)(x - z_2)\ldots(x - z_n)$$

where a is the leading coefficient of $f(x)$ and $z_1, z_2, \ldots, z_n$ are the complex zeros of $f(x)$. The z_i are not necessarily distinct numbers; some may be repeated.

<u>Fundamental Polynomial Connections in the Complex Case:</u> The following statements about polynomial function f are equivalent if k is a complex number:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$.
2. k is a zero of the function f.
3. $x - k$ is a factor of $f(x)$.

Examples:

Write the polynomial function in standard form, and identify the zeros of the function and the x-intercepts of its graph.

a) $f(x) = (x - 2i)(x + 2i) = x^2 + 2ix - 2ix - 4i^2 = x^2 + 4$, it has two zeros at $x = 2i$ and $x = -2i$, because the zeros are not real there are no x-intercepts.

b) $f(x) = (x - 5)(x - \sqrt{2}i)(x + \sqrt{2}i) = (x - 5)(x^2 + 2) = x^3 - 5x^2 + 2x - 10$, there are three zeros at $x = 5$, $x = \sqrt{2}i$, and $x = -\sqrt{2}i$. Only $x = 5$ is a real root so there is just one x-intercept at 5.

<u>Complex Conjugate Zeros:</u>

Suppose that $f(x)$ is a polynomial function with real coefficients. If a and b are real numbers with $b \neq 0$ and ***a + bi* is a zero** of $f(x)$, **then** its complex conjugate ***a - bi* is also a zero** of $f(x)$.

Example: Finding a polynomial given zeros.

Write a polynomial function in standard form with real coefficients whose zeros include -2, 1, $1 - 2i$.

If $1 - 2i$ is a zero, then $1 + 2i$ is also a zero. So, the factors we get from the zeros are $(x + 2)(x - 1)[x - (1 - 2i)][x - (1 + 2i)]$.

Therefore, $f(x) = (x + 2)(x - 1)[x - (1 - 2i)][x - (1 + 2i)]$

$$= (x^2 + x - 2)(x - 1 + 2i)(x - 1 - 2i)$$

$$= (x^2 + x - 2)(x^2 - x - 2ix - x + 1 + 2i + 2ix - 2i - 4i^2)$$

$$= (x^2 + x - 2)(x^2 - 2x + 5) = x^4 - x^3 + x^2 + 9x - 10$$

Finding complex zeros:

The complex number i is a zero of $f(x) = x^4 - 3x^2 - 4$, find the remaining zeros of $f(x)$ and write it in its linear factorization.

Using synthetic division we can use i to show that $f(x - i) = 0$.

$$\begin{array}{r|rrrrr} i & 1 & 0 & -3 & 0 & -4 \\ & & i & -1 & -4i & 4 \\ \hline & 1 & i & -4 & -4i & 0 \end{array}$$

the complex conjugate of i is $-i$ so,

$$\begin{array}{r|rrrr} -i & 1 & i & -4 & -4i \\ & & -i & 0 & 4i \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

From the last row we get $x^2 - 4$, which factors to $(x + 2)(x - 2)$.

We can now write the linear factorization as $f(x) = (x + 2)(x - 2)(x - i)(x + i)$.

Every polynomial function with real coefficients can be written as a product of linear factors and irreducible factors, each with real coefficients.

Every polynomial function of odd degree with real coefficients has at least one real zero.

Factoring a Polynomial:

Write $f(x) = 3x^5 - 2x^4 + 6x^3 - 4x^2 - 24x + 16$ as a product of linear and irreducible quadratic factors, each with real coefficients.

Solution:

Using the Rational Zeros Theorem will give possible rational zeros of ***f***.

$$\frac{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16}{\pm 1, \pm 3}$$

Graphing the function will help to identify which possible zeros to try first. Using synthetic division we find that $x = \frac{2}{3}$ is a zero. We can then write the factorization using the zero and the bottom line of the synthetic division.

We get

$$\begin{aligned} f(x) &= (x - \tfrac{2}{3})(3x^4 + 6x^2 - 24) \\ &= (x - \tfrac{2}{3})(3)(x^4 + 2x^2 - 8) \\ &= (3x - 2)(x^2 - 2)(x^2 + 4) \\ &= (3x - 2)(x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \end{aligned}$$

Because the zeros of $x^2 + 4$ are complex, any further factorization would introduce nonreal complex coefficients. We have taken the factorization as far as possible, subject to the condition that each factor has real coefficients.

Polynomial Function of Odd Degree:

Every polynomial function of odd degree with real coefficients has at least one real zero.

Why?